

Qualitative evolution in $f(R)$ cosmologies

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Abstract

We investigate the qualitative evolution of $(D + 1)$ -dimensional cosmological models in $f(R)$ gravity for the general case of the function $f(R)$. The analysis is specified for various examples, including the $(D + 1)$ -dimensional generalization of the Starobinsky model, models with polynomial and exponential functions. The cosmological dynamics are compared in the Einstein and Jordan representations of the corresponding scalar-tensor theory. The features of the cosmological evolution are discussed for Einstein frame potentials taking negative values in certain regions of the field space.

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1 Introduction

Recent observations of the cosmic microwave background, large scale structure and type Ia supernovae have provided strong evidence that at present epoch the expansion of the universe is accelerating [1]. Assuming that General Relativity correctly describes the large scale dynamics of the Universe, this means that the energy density is currently dominated by a form of energy having negative pressure. This type of a gravitational source is referred as dark energy. The simplest model for the latter, consistent with all observations to date, is a cosmological constant. From the cosmological point of view, a cosmological constant is equivalent to the vacuum energy in quantum field theory. However, the value of a cosmological constant inferred from cosmological observations is many orders of magnitude smaller than the value one might expect based on quantum field-theoretical considerations. This large discrepancy is one of the motivations to consider alternative models for dark energy. To account for the missing energy density, instead of a cosmological constant one could add a new component of matter, such as quintessence (see [2] for a review). The latter is modeled by slow-rolling scalar fields. However, because of very small mass of a scalar field responsible for the acceleration, it is generally difficult to construct viable potentials on the base of particle physics.

More recently, it has been shown that suitable modifications of General Relativity can result in an accelerating expansion of the Universe at present epoch. These modifications fall into two general groups. The first one consists of scalar-tensor theories that are most widely considered extensions of General Relativity [3]. In addition to the metric tensor, these theories contain scalar fields in their gravitational sector and typically arise in the context of models with extra dimensions (Kaluza-Klein-type models, braneworld scenario) and within the framework of the low-energy string effective gravity. In the second group of models, the Ricci scalar R in the Einstein-Hilbert action is replaced by a general function $f(R)$ (for recent reviews see [4]-[10]). One of the first models for inflation with quadratic in the Ricci scalar Lagrangian, proposed by Starobinsky [11], falls into this class of theories. An additional motivation for the $f(R)$ theories comes from quantum field theory in classical curved

backgrounds [12] and from string theories. The recent investigations of cosmological models in the $f(R)$ theories of gravity have shown a possibility for a unified description of the inflation and the late-time acceleration.

$f(R)$ gravities can be recast as scalar tensor theories of a special type with a potential determined by the form of the function $f(R)$. Various special forms of this function have been discussed in the literature. In particular, the functions were considered that realize the cosmological dynamics with radiation dominated, matter dominated and accelerated epoch. Unified models of inflation and dark energy have been studied as well [8]. In the present paper we consider the qualitative evolution of the cosmological model for a general $f(R)$ function. The general analysis is specified for various examples, including the original Starobinsky model. We have organized the paper as follows. In the next section we present the action of $f(R)$ gravity in the form of the action of a scalar-tensor theory in a general conformal representation. Then, the general action is specified for the Einstein frame with a scalar field having a canonical kinetic term. The specific form of the scalar field potential is given for various examples of the $f(R)$ function. The corresponding cosmological model is described in section 3 and the relations between the functions in the Einstein and Jordan frames are discussed. The qualitative analysis of the spatially flat gravi-scalar model is presented in section 4. The phase portraits are plotted for special cases. The main results of the paper are summarized in section 5.

2 $f(R)$ gravity as a scalar-tensor theory: Conformal representations and examples

The action in $(D + 1)$ -dimensional $f(R)$ theory of gravity has the form

$$S = \int d^{D+1}x \sqrt{|g|} [f(R) + L_m(g_{ik}, \psi)], \quad (1)$$

where $L_m(g_{ik}, \psi)$ is the Lagrangian density for non-gravitational matter collectively denoted by ψ . It is well known (see [4]-[10]) that (1) can be presented in the form of the action for scalar-tensor gravity. In order to show that we consider the action

$$S = \int d^{D+1}x \sqrt{|g|} [f'(\eta)(R - \eta) + f(\eta) + L_m(g_{ik}, \psi)], \quad (2)$$

with a scalar field η . The equation for the latter is reduced to $f''(\eta)(R - \eta) = 0$. Assuming that $f''(\eta) \neq 0$, from the field equation we get $\eta = R$. With this solution, the action (2) is reduced to the original action (1).

Introducing a new scalar field φ in accordance with

$$\varphi = -f'(\eta), \quad (3)$$

the action (2) is written in the form

$$S = \int d^{D+1}x \sqrt{|g|} [-\varphi R - V(\varphi) + L_m(g_{ik}, \psi)], \quad (4)$$

with the scalar potential

$$V(\varphi) = -f(\eta(\varphi)) - \varphi\eta(\varphi). \quad (5)$$

Here, we have assumed that the function $\eta(\varphi)$, defined by (3), is invertible. The action (4) describes a scalar-tensor theory. In the representation (4) the Lagrangian density of the non-gravitational matter does not depend on the scalar field φ . Hence, the representation corresponds to the Jordan frame.

The scalar-tensor theories can be presented in various representations which are related by conformal transformations of the metric tensor. Let us consider a general conformal transformation

$$g_{ik} = \Omega^2(\varphi)\tilde{g}_{ik}, \quad (6)$$

with a sufficiently smooth function $\Omega(\varphi)$. Up to total derivative terms, in the new conformal representation the action takes the form

$$S = \int d^{D+1}x \sqrt{|\tilde{g}|} \left[-\tilde{F}_R(\varphi)\tilde{R} + \tilde{F}_\varphi(\varphi)\tilde{g}^{ik}\partial_i\varphi\partial_k\varphi - \tilde{V}(\varphi) + \tilde{L}_m(\varphi, \tilde{g}_{ik}, \psi) \right], \quad (7)$$

where we have introduced the notations

$$\begin{aligned} \tilde{F}_\varphi(\varphi) &= -D\Omega^{D-1}(\Omega'/\Omega) [(D-1)\varphi\Omega'/\Omega + 2], \\ \tilde{F}_R(\varphi) &= \Omega^{D-1}\varphi, \quad \tilde{V}(\varphi) = \Omega^{D+1}V(\varphi), \quad \tilde{L}_m(\varphi, \tilde{g}_{ik}, \psi) = \Omega^{D+1}L_m(\Omega^2\tilde{g}_{ik}, \psi), \end{aligned} \quad (8)$$

and the prime stands for the derivative with respect to φ . Qualitative evolution of the models of the type (7) with $\tilde{V}(\varphi) = 0$, arising in higher-loop string cosmology, has been discussed in [13].

By choosing the conformal factor as

$$\Omega(\varphi) = \Omega_E(\varphi) = m_P\varphi^{1/(1-D)}, \quad (9)$$

where $m_P = 1/(16\pi G_{D+1})^{1/(D-1)}$ is the Planck mass in $(D+1)$ -dimensions and G_{D+1} is the corresponding gravitational constant, we get $\tilde{F}_R(\varphi) = m_P^{D-1}$. In the corresponding conformal frame, referred as the Einstein frame, the gravitational part of the action takes the form of that for $(D+1)$ -dimensional General Relativity:

$$S = \int d^{D+1}x \sqrt{|g_{(E)}|} \left[-m_P^{D-1}R_{(E)} + F_\varphi^{(E)}(\varphi)g_{(E)}^{ik}\partial_i\varphi\partial_k\varphi - V_{(E)}(\varphi) + L_m^{(E)}(\varphi, g_{(E)ik}, \psi) \right]. \quad (10)$$

In the Einstein frame, the function in front of the scalar field kinetic term and the non-gravitational Lagrangian density are related to the functions in the original action by the formulae

$$F_\varphi^{(E)}(\varphi) = m_P^{D-1}\frac{D\varphi^{-2}}{D-1}, \quad V_{(E)}(\varphi) = \frac{m_P^{D+1}V(\varphi)}{\varphi^{(D+1)/(D-1)}}, \quad (11)$$

and

$$L_m^{(E)}(\varphi, g_{(E)ik}, \psi) = m_P^{D+1}\varphi^{(D+1)/(1-D)}L_m(\Omega_E^2g_{(E)ik}, \psi). \quad (12)$$

The function $F_\varphi^{(E)}(\varphi)$ has a pole at $\varphi = 0$. As it has been discussed in [14], the presence of singularities in the kinetic function for a scalar field provides an additional mechanism for the cosmological stabilization of scalar fields. Note that in a large class of models discussed in [15] the inflationary predictions for the spectral index and for the tensor-to-scalar ratio are determined by the leading terms in the Laurent expansions of the functions $F_\varphi^{(E)}(\varphi)$ and $V_{(E)}(\varphi)$.

Introducing a new scalar field ϕ according to the relation

$$\phi = \phi_0 \ln(\varphi/\varphi_0), \quad (13)$$

with φ_0 being an integration constant and

$$\phi_0 = m_P^{(D-1)/2} \sqrt{\frac{2D}{D-1}}, \quad (14)$$

the kinetic term for the scalar field is written in the standard canonical form:

$$S = \int d^Dx \sqrt{|g_{(E)}|} \left[-m_P^{D-1}R_{(E)} + \frac{1}{2}g_{(E)}^{ik}\partial_i\phi\partial_k\phi - V_E(\phi) + L_m^E(\phi, g_{(E)ik}, \psi) \right]. \quad (15)$$

Here, the non-gravitational Lagrangian density is expressed as

$$L_m^E(\phi, g_{(E)ik}, \psi) = L_m^{(E)}(\varphi_0 e^{\phi/\phi_0}, g_{(E)ik}, \psi). \quad (16)$$

Note that in the Einstein representation we have a direct interaction between the non-gravitational matter and the scalar field.

In what follows it is convenient to take the integration constant in (13) $\varphi_0 = m_P^{D-1}$. With this choice, the Einstein frame potential in terms of the canonical scalar field takes the form

$$V_E(\phi) = -\exp\left(-\frac{D+1}{D-1}\frac{\phi}{\phi_0}\right) [f(\eta(\varphi)) + \varphi\eta(\varphi)], \quad (17)$$

where

$$\varphi = m_P^{D-1} e^{\phi/\phi_0}, \quad (18)$$

and the function $\eta(\varphi)$ is obtained by inverting of (3). In the qualitative analysis described below we need also to have the first and second derivatives of the potential. From (17) we can obtain the expressions

$$\frac{dV_E(\phi)}{d(\phi/\phi_0)} = \exp\left(-\frac{D+1}{D-1}\frac{\phi}{\phi_0}\right) f(\eta(\varphi)) - \frac{2}{D-1}V_E(\phi), \quad (19)$$

for the first derivative and

$$\frac{d^2V_E(\phi)}{d(\phi/\phi_0)^2} = \exp\left(-\frac{D+1}{D-1}\frac{\phi}{\phi_0}\right) \left[\frac{\varphi^2}{f''(\eta(\varphi))} - \frac{D+3}{D-1}f(\eta(\varphi)) \right] + \frac{4V_E(\phi)}{(D-1)^2}, \quad (20)$$

for the second derivative.

Let us consider the form of the potential $V_E(\phi)$ for some examples of the function $f(R)$. A number of specific choices for this function have been discussed in the literature. In the models with quantum corrections to the Einstein-Hilbert Lagrangian the function $f(R)$ is of the polynomial form. A similar structure is obtained in the string-inspired models with the effective action expanded in powers of the string tension. However, it should be noted that in both these types of models coming from high-energy physics, the Lagrangian density in addition to the scalar curvature contains other scalars constructed from the Riemann tensor. In this context, the $f(R)$ theories can be considered as models simple enough to be easy to handle from which we gain some insight in modifications of gravity. In some models proposed for dark energy the function $f(R)$ contains terms with the inverse power of the Ricci scalar. For one of the first models of this type $f(R) = m_P^{D-1}(-R + \gamma/R^m)$ with γ and $m > 0$ being constants [16]. However, there is a matter instability problem in these models. The model with an additional term βR^2 in the brackets has been discussed in [17]. Models containing in $f(R)$ exponential functions of the form $e^{\gamma R}$ and providing the accelerating cosmological solutions without a future singularity are considered in [18]. Examples of the $f(R)$ functions, containing combinations of the powers and exponentials of R , that allow to construct models with a late-time accelerated expansion consistent with local gravity constraints, are studied in references [19] (see also [4]-[10]). For example, in the Tsujikawa model $f(R) = m_P^{D-1}[-R + \gamma \tanh(R/R_0)]$, whereas in the Hu and Sawicki model $f(R) = m_P^{D-1}[-R + \gamma(1 + (R/R_0)^{-m})]$ with constants γ and R_0 .

We start our discussion with a $(D+1)$ -dimensional generalization of the Starobinsky model (see [20] for the discussion of inflation in this type of models). The corresponding lagrangian density for the gravitational field is taken as

$$f(R) = m_P^{D-1} (-R + \beta R^2), \quad (21)$$

where β is a constant. The potential in terms of the canonical scalar field is written in the form

$$V_E(\phi) = V_{E0} \exp\left(\frac{D-3}{D-1}\frac{\phi}{\phi_0}\right) \left(1 - e^{-\phi/\phi_0}\right)^2, \quad (22)$$

where $V_{E0} = m_P^{D-1}/(4\beta)$. For $\beta > 0$ and $\phi \neq 0$ the potential (22) is positive. It has a minimum at $\phi = 0$ with $V_E(0) = 0$. In figure 1 we have plotted the potential (22) as a function of ϕ/ϕ_0 for various

values of the spatial dimension (numbers near the curves). As is seen from the graphs, in the case $D = 3$ an inflationary plateau appears for large values of ϕ/ϕ_0 which corresponds to the Starobinsky inflation (for a recent discussion of the universality of the inflation in the Starobinsky model and its generalizations see [21]). Hence, from the point of view of the Starobinsky inflation, the spatial dimension $D = 3$ is special.

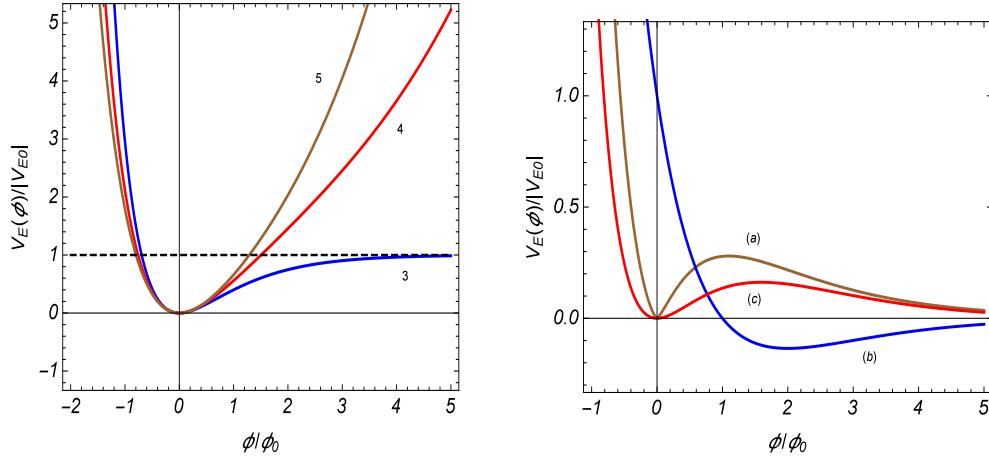


Figure 1: The left panel presents the Einstein frame potentials in the Starobinsky model for different values of the spatial dimension (numbers near the curves). On the right panel the potentials corresponding to the $f(R)$ functions (28) (for $n = 4$, curve (a)), (32) (curve (b)) and (35) (curve (c)) are plotted.

As the next example, consider the model with a polynomial function

$$f(R) = m_P^{D-1} \left(-R + \sum_{l=2}^n \beta_l R^l \right), \quad (23)$$

with $n \geq 2$. For the Einstein frame potential one gets the expression

$$V_E(\phi) = m_P^{D-1} \exp \left(-\frac{D+1}{D-1} \frac{\phi}{\phi_0} \right) \sum_{l=2}^n (l-1) \beta_l \chi^l(\phi), \quad (24)$$

where the function $\chi(\phi)$ is defined by the relation

$$\sum_{l=2}^n l \beta_l \chi^{l-1}(\phi) = 1 - e^{\phi/\phi_0}. \quad (25)$$

The relations (24) and (25) define the potential $V_E(\phi)$ in the parametric form with χ being the parameter.

Let us investigate the asymptotics of the potential (24) in the limits $\phi \rightarrow \pm\infty$. From (25) it follows that in the limit $\phi \rightarrow +\infty$ one has $\chi(\phi) \approx [-e^{\phi/\phi_0}/(n\beta_n)]^{1/(n-1)}$. In particular, we see that for an odd n one should have $\beta_n < 0$. For the asymptotic behavior of the potential we get

$$V_E(\phi) \approx -\frac{(n-1)m_P^{D-1}}{n^{n/(n-1)}(-\beta_n)^{1/(n-1)}} \exp \left[\left(\frac{1}{n-1} - \frac{2}{D-1} \right) \frac{\phi}{\phi_0} \right]. \quad (26)$$

Hence, in the limit $\phi \rightarrow +\infty$ one has $V_E(\phi) \rightarrow 0$ for $n > (D+1)/2$. For $n = (D+1)/2$ the potential tends to the finite limiting value determined by the coefficient of the exponent in (26). In the case

$n < (D+1)/2$, the potential tends to $+\infty$ or $-\infty$ depending on the sign of the coefficient β_n . In the limit $\phi \rightarrow -\infty$ the function $\chi(\phi)$ tends to the finite limiting value $\chi_- \equiv \chi(-\infty)$ determined by the relation $\sum_{l=2}^n l\beta_l \chi_-^{l-1} = 1$ (see (25)). As a result, the potential behaves as

$$V_E(\phi) \approx m_P^{D-1} \exp\left(-\frac{D+1}{D-1} \frac{\phi}{\phi_0}\right) \sum_{l=2}^n (l-1)\beta_l \chi_-^l, \quad (27)$$

for $\phi \rightarrow -\infty$. As is seen, the functional form of the potential in this region is universal and the information on the coefficients of the polynomial function (23) is contained in the coefficient only.

As a special case of (23), let us consider the model

$$f(R) = m_P^{D-1} (-R + \beta_n R^n), \quad (28)$$

with even n and $\beta_n > 0$. The corresponding potential is nonnegative and is given by the expression:

$$V_E(\phi) = V_{E0} \exp\left(-\frac{D+1}{D-1} \frac{\phi}{\phi_0}\right) [(1 - e^{\phi/\phi_0})^n]^{1/(n-1)}, \quad (29)$$

where

$$V_{E0} = \frac{(n-1)m_P^{D-1}}{n^{n/(n-1)}\beta_n^{1/(n-1)}}. \quad (30)$$

For $n = 2$ the potential is reduced to the one for the Starobinsky model. In the limit $\phi/\phi_0 \gg 1$ the potential behaves as

$$V_E(\phi) \propto \exp\left[\left(\frac{1}{n-1} - \frac{2}{D-1}\right) \frac{\phi}{\phi_0}\right]. \quad (31)$$

Hence, in this limit one has $V_E(\phi) \rightarrow +\infty$ for $n < (D+1)/2$ and $V_E(\phi) \rightarrow 0$ for $n > (D+1)/2$. For $n = (D+1)/2$, in the limit $\phi/\phi_0 \rightarrow +\infty$ the potential has a nonzero plateau: $V_E(\phi) \rightarrow V_{E0}$. The potential (29) for $n = 4$ and $D = 3$ is depicted in the right panel of figure 1 (graph (a)).

For the next example we take the function

$$f(R) = f_0 e^{\gamma R}. \quad (32)$$

The corresponding potential takes the form

$$V_E(\phi) = V_{E0} \exp\left(-\frac{2}{D-1} \frac{\phi}{\phi_0}\right) \left[\phi/\phi_0 + \ln(-M_{D+1}^{D-1}/\gamma f_0) - 1\right], \quad (33)$$

with $V_{E0} = -m_P^{D-1}/\gamma$. For $|\gamma R| \ll 1$ one has $f(R) = f_0 + f_0 \gamma R$. Taking $f_0 \gamma = -m_P^{D-1}$, the linear in R term coincides with the Hilbert-Einstein lagrangian density. With this choice the potential simplifies to

$$V_E(\phi) = V_{E0} \exp\left(-\frac{2}{D-1} \frac{\phi}{\phi_0}\right) (\phi/\phi_0 - 1), \quad (34)$$

The graph of this potential for $\gamma > 0$ is plotted in the right panel of figure 1 (curve (b)). The value of the potential at the minimum is negative. Cosmological consequences of this feature will be discussed below. Note that in this case $f_0 = -V_{E0} > 0$ and for $|\gamma R| \ll 1$ the model reduces to General Relativity with a negative cosmological constant V_{E0} .

In the case of the function

$$f(R) = f_0 (e^{\gamma R} - 1), \quad (35)$$

with $f_0 \gamma = -m_P^{D-1}$ and for small curvatures, corresponding to $|\gamma R| \ll 1$, the model is reduced to General Relativity with zero cosmological constant. The corresponding potential is given by the expression

$$V_E(\phi) = V_{E0} \exp\left(-\frac{D+1}{D-1} \frac{\phi}{\phi_0}\right) \left[1 + e^{\phi/\phi_0} (\phi/\phi_0 - 1)\right], \quad (36)$$

with the same notation V_{E0} as in (34). This potential for $\gamma < 0$ ($V_{E0} > 0$) is plotted in figure 1 (curve (c)).

3 Cosmological model

In this section we consider a homogeneous and isotropic cosmological model described by the Einstein frame action (15). The corresponding line element has the form

$$ds_E^2 = dt^2 - a^2(t)dl^2 \quad (37)$$

where dl is the line element of a D - dimensional space of constant curvature, $a(t)$ is the scale factor. From the homogeneity of the model it follows that the scalar field should also depend on time only, $\phi = \phi(t)$. From the field equations we obtain that the energy-momentum tensor corresponding to the metric (37) is diagonal and can be presented in the perfect fluid form $T_i^k = \text{diag}(\varepsilon, \dots, -p, \dots)$, where ε is the energy density and p is the effective pressure.

For a model with a flat space, the Einstein frame evolution equations for the scale factor and the scalar field can be written as

$$\begin{aligned} \dot{H} + DH^2 &= \frac{m_P^{1-D}}{D-1} \left[\frac{1-w}{2} \varepsilon + V_E(\phi) \right], \\ \ddot{\phi} + DH\dot{\phi} &= \alpha\varepsilon - V_E'(\phi), \\ D(D-1)H^2 &= m_P^{1-D} \left[\varepsilon + \dot{\phi}^2/2 + V_E(\phi) \right], \end{aligned} \quad (38)$$

where the overdot denotes the time derivative and the following notations are introduced

$$H = \frac{\dot{a}}{a}, \quad w = \frac{p}{\varepsilon}, \quad \alpha = \frac{1}{\varepsilon \sqrt{|g(E)|}} \frac{\delta L_m^E \sqrt{|g(E)|}}{\delta \phi}. \quad (39)$$

Excluding H by using the last equation of (38) and introducing dimensionless quantities $x = \phi/\phi_0$, $\tau = t/t_0$, with t_0 being a positive constant with the dimension of time, for expanding models the set of cosmological equations is written in terms of the third order autonomous dynamical system

$$\begin{aligned} \frac{dx}{d\tau} &= y, \\ \frac{dy}{d\tau} &= -by[2\epsilon + y^2 + 2V(x)]^{1/2} + \phi_0\alpha\epsilon - V'(x), \\ \frac{d\epsilon}{d\tau} &= -\{b(1+w)[2\epsilon + y^2 + 2V(x)]^{1/2} + \phi_0\alpha y\}\epsilon. \end{aligned} \quad (40)$$

Here we have defined dimensionless functions

$$V(x) = (t_0/\phi_0)^2 V_E(\phi_0 x), \quad \epsilon = (t_0/\phi_0)^2 \varepsilon, \quad (41)$$

and

$$b = \frac{D}{D-1}. \quad (42)$$

Note that the function $\phi_0\alpha$ is dimensionless as well. The Einstein frame Hubble function is expressed in terms of the variables of the dynamical system (40) as

$$H^2 = \frac{2\epsilon + y^2 + 2V(x)}{(D-1)^2 t_0^2}. \quad (43)$$

The set of equations (40) describes the cosmological dynamics in the Einstein frame. The corresponding dynamics in the Jordan frame is obtained by using the conformal transformation (6) with the function (9). For the line element in the Jordan frame one has $ds_J^2 = dt_J^2 - a_J^2(t_J)dl^2$, where

the comoving time coordinate and the scale factor are related to the corresponding Einstein frame quantities by

$$dt_J = m_P \varphi^{1/(1-D)} dt, \quad a_J(t_J) = m_P \varphi^{1/(1-D)} a(t). \quad (44)$$

From here we get the relation between the Hubble functions in the Einstein and Jordan frames:

$$H_J(t_J) = \frac{1}{a_J(t_J)} \frac{da_J(t_J)}{dt_J} = \frac{\varphi^{1/(D-1)}}{m_P} \left[H(t) - \frac{\dot{\phi}/\phi_0}{D-1} \right]. \quad (45)$$

Substituting the expression for $H(t)$ from the last equation of (38), this gives

$$H_J = \frac{\varphi^{1/(D-1)}/\phi_0}{(D-1)m_P} \left[\pm \sqrt{2\varepsilon + \dot{\phi}^2 + 2V_E(\phi)} - \dot{\phi} \right], \quad (46)$$

where the upper/lower sign corresponds to expanding/contracting models in the Einstein frame. From the relation (46) it follows that for $V_E(\phi) + \varepsilon > 0$ the expansion/contraction in the Einstein frame corresponds to the expansion/contraction in the Jordan frame.

4 Qualitative analysis of gravi-scalar models

The dynamical system (40) has an invariant phase plane $\varepsilon = 0$ which corresponds to the pure gravi-scalar models. First we consider the qualitative dynamics of these models (for applications of the qualitative theory of dynamical systems in cosmology see [22]).

4.1 General analysis

In what follows it is convenient to introduce dimensionless quantities $x = \phi/\phi_0$, $\tau = t/t_0$, where t_0 is a positive constant with the dimension of time. In terms of these variables, for pure gravi-scalar models the system (40) is reduced to the following second order dynamical system

$$\begin{aligned} \frac{dx}{d\tau} &= y, \\ \frac{dy}{d\tau} &= -by[y^2 + 2V(x)]^{1/2} - V'(x), \end{aligned} \quad (47)$$

where $\dot{\phi} = (\phi_0/t_0)y$ and

$$b = c\phi_0 = \frac{D}{D-1}, \quad V(x) = (t_0/\phi_0)^2 V_E(\phi_0 x). \quad (48)$$

For Einstein frame expanding models, the Hubble function is expressed in terms of the solution of dynamical system (47) as

$$H = \frac{\sqrt{y^2 + 2V(x)}}{(D-1)t_0}. \quad (49)$$

For nonnegative potentials, introducing the function $X(x)$ in accordance with the relation $y = \sqrt{2V(x)} \sinh X(x)$, the equation for the phase trajectories is written as

$$X'(x) = -b - \frac{V'(x)}{2V(x)} \coth X(x), \quad (50)$$

This equation is exactly solvable in a special case of exponential potentials (for a recent discussion of scalar cosmologies with exponential potentials see [23]):

$$V(x) = V_1 e^{\sigma x}, \quad (51)$$

with V_1 and σ being constants. The equation of the phase trajectories is written in the parametric form as

$$\begin{aligned} x &= \frac{1}{2b} \left[\frac{2q}{1-q^2} \ln|1+qz| + \frac{\ln|z-1|}{1+q} + \frac{\ln|z+1|}{q-1} \right] + C, \\ y &= \sqrt{2V_1} e^{\sigma x/2} \frac{\text{sgn}(z)}{\sqrt{z^2-1}}, \end{aligned} \quad (52)$$

where $q = \sigma/(2b)$, $z = \coth X$ and C is an integration constant. The corresponding Hubble function is found from (49):

$$H = \frac{\sqrt{2V_1} e^{\sigma x/2} |z|}{(D-1)t_0 \sqrt{z^2-1}}. \quad (53)$$

It can be seen that the limit $z^2 \rightarrow 1$ corresponds to the early stages of the cosmological expansion ($\tau \rightarrow 0$). In this limit one has $y^2 \gg V(x)$ and the cosmological dynamics is dominated by the kinetic energy of the scalar field. Under the condition $|\sigma| < 2b$, the limit $z \rightarrow -1/q$ corresponds to the late stages of the expansion, $\tau \rightarrow +\infty$. In this limit the kinetic and potential energies of the scalar field are of the same order: $y^2 \approx 2V(x)/(q^{-2}-1)$.

For $|\sigma| < 2b$ the equation (50) has a special solution $\coth X = -1/q$. The corresponding phase trajectory is described by the equation

$$y = -\frac{\sigma \sqrt{2V_1} e^{\sigma x/2}}{\sqrt{4b^2 - \sigma^2}}. \quad (54)$$

Note that for this solution the ratio of the kinetic and potential energies of the scalar field is a constant. For the ratio of the corresponding pressure and energy density one gets $p_\phi/\varepsilon_\phi = \sigma^2/(2b^2) - 1$. The time dependence of the special solution is given by

$$\begin{aligned} x &= x_1 - \frac{2}{\sigma} \ln \tau, \\ a(t) &= \text{const } (t/t_0)^{\beta_E}, \quad \beta_E = \frac{4D}{(D-1)^2 \sigma^2}, \end{aligned} \quad (55)$$

where

$$x_1 = \left(2 \frac{4b^2/\sigma^2 - 1}{\sigma^2 B} \right)^{1/\sigma}. \quad (56)$$

For $|\sigma| < 2\sqrt{D}/(D-1)$, the expansion described by (55) corresponds to a power-law inflation in the Einstein frame. The special solution (55) is a future attractor ($t \rightarrow +\infty$) for a general solution (52).

Now we turn to the qualitative analysis of the system (47) for the general case of the potential $V(x)$ (for applications of the qualitative theory of dynamical systems in cosmology see [22]). The critical points for the system are the points of the phase space (x, y) with the coordinates $(x_c, 0)$ where $V'(x_c) = 0$, $V(x_c) \geq 0$. For the corresponding solution the Hubble function is a constant, $H = H_c$, with

$$H_c^2 = \frac{V_E(\phi_c)}{m_P^{D-1} D(D-1)}. \quad (57)$$

This solution describes the Minkowski spacetime for $V_E(\phi_c) = 0$ and the de Sitter spacetime for $V_E(\phi_c) > 0$. In the latter case for the cosmological constant one has $\Lambda = V_E(\phi_c)/(2m_P^{D-1})$.

The character of the critical points is defined by the eigenvalues

$$\lambda_{1,2} = -b\sqrt{V_c/2} \pm \sqrt{b^2 V_c/2 - V_c''}, \quad (58)$$

where $V_c = V(x_c)$, $V_c'' = V''(x_c)$. From here it follows that for $V_c'' < 0$ (x_c is a maximum of the potential $V(x)$) the critical point is a saddle. The directions of the corresponding separatrices are

determined by the unit vectors $\mathbf{n}^{(i)} = (1, \lambda_i)/\sqrt{1 + \lambda_i^2}$, $i = 1, 2$. For $V_c'' > 0$ (x_c is a minimum of the potential $V(x)$) two cases should be considered separately. When $0 < V_c'' < b^2 V_c/2$, the critical point is a stable node. For $0 < b^2 V_c/2 < V_c''$ the critical point is a stable sink. In the case $V_c > 0$, $V_c^{(i)} \equiv (d^i V/dx^i)_{x=x_c} = 0$ for $i = 1, \dots, n-1$, and $V_c^{(n)} \neq 0$, the critical point is (i) a saddle for even n and $V_c^{(n)} < 0$, (ii) a stable node for even n and $V_c^{(n)} > 0$, (iii) a degenerate critical point with one stable node sector and with two saddle sectors for odd n . Another degenerate case corresponds to $V_c = 0$ and $V_c'' > 0$. In this case the critical point is a stable sink.

By using the expressions (19) and (20), we can express V_c and V_c'' in terms of the function $f(R)$:

$$\begin{aligned} V_c &= \frac{D-1}{2} \exp\left(-\frac{D+1}{D-1}x_c\right) f_c, \\ V_c'' &= \exp\left(\frac{D-3}{D-1}x_c\right) \frac{m_P^{2(D-1)}}{f_c''} - \frac{2(D+1)}{(D-1)^2} V_c, \end{aligned} \quad (59)$$

where

$$f_c = f(\eta(\varphi_c)), \quad f_c'' = f''(\eta(\varphi_c)), \quad (60)$$

and $\varphi_c = m_P^{D-1} e^{x_c}$. Note that $R_c = \eta(\varphi_c)$ corresponds to the Ricci scalar in the original representation (1), evaluated at the critical point. By taking into account that $V_c \geq 0$, from (59) we conclude that the critical points correspond to the values of the Ricci scalar for which $f(R_c) \geq 0$. We also see that $V_c'' < 0$ for $f_c'' < 0$ and, hence, in this case the critical point is unstable being a saddle point.

We should also consider the behavior of the phase trajectories at the infinity of the phase plane. With this aim, it is convenient to introduce polar coordinates (ρ, θ) defined as

$$x = \frac{\rho \cos \theta}{1 - \rho}, \quad y = \frac{\rho \sin \theta}{1 - \rho}, \quad (61)$$

with $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$. Now the phase space is mapped onto a unite circle. The points at infinity correspond to $\rho = 1$. For the potentials having the asymptotic behavior $V(x) \sim B|x|^m$, $m < 4$, in the limit $x \rightarrow \infty$ one has the following critical points on the circle $\rho = 1$. The points $\theta = 0$ and $\theta = \pi$ are stable nodes for $m < 0$ and saddles with two sectors for $m > 0$. In the latter case the sectors are separated by a special solution described by the trajectory

$$y(x) \approx -\frac{V'(x)}{b\sqrt{2V(x)}} \sim \frac{m}{b} \sqrt{B/2} |x|^{m/2-1}, \quad (62)$$

for $|x| \rightarrow \infty$. In the vicinity of the points $\theta = \pi/2$ and $\theta = 3\pi/2$ the potential terms can be neglected and these points are unstable degenerate nodes. For $m = 4$ the nature of the critical points at $\theta = \pi/2$ and $\theta = 3\pi/2$ remains the same. In this case the other critical points correspond to $\theta = -\arctan(\sqrt{8B}/b)$ and $\theta = \pi - \arctan(\sqrt{8B}/b)$. The phase portrait near these points have two saddle sectors which are separated by the trajectory corresponding to the special solution (62). For $m > 4$ there are two critical points on the circle $\rho = 1$ corresponding to $\theta = \pi/2$ and $\theta = 3\pi/2$. These points are degenerate and have an unstable node sector and a saddle sector separated by the special solution (62). Similar behavior of the phase trajectories at the infinity takes place for the potentials with the asymptotic behavior $V(x) \sim B e^{\sigma|x|}$, $\sigma > 0$, for $x \rightarrow \infty$ and for the values of the parameter $0 < \sigma < 2b$. The separatrix between the saddle and node sectors is described by the special solution $y \approx -\text{sgn}(x)\sigma\sqrt{2B}e^{\sigma|x|/2}/\sqrt{4b^2 - \sigma^2}$ for $x \rightarrow \infty$. The general solution behaves as $y \approx -\text{sgn}(x)C e^{b|x|}$, with a positive constant C . This behavior coincides with that in the absence of the potential. For $\sigma \geq 2b$ the dynamical system (47) has no critical points at infinity (on the circle $\rho = 1$).

We have described the general evolution of gravi-scalar models. In the presence of a barotropic non-gravitational matter with $w = \text{const}$ the evolution is described by the three-dimensional dynamical

system (40) with the phase space (x, y, ϵ) . Note that, as a consequence of the term $\phi_0 \alpha y$ in the equation for $d\epsilon/d\tau$, in expanding models the energy density, in general, is not a monotonically decreasing function of the Einstein frame time coordinate for $w > -1$. The phase trajectories corresponding to gravi-scalar models lie in the plane $(x, y, 0)$ that forms an invariant subspace. The points $(x_c, 0, 0)$ of this subspace are critical points of the system (47). For the corresponding eigenvalues one has $(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_{1,2}$ are given by the expression (58) and $\lambda_3 = -b(1+w)\sqrt{2V_c}$. Notice that these eigenvalues do not depend on the function α . We see that for $w > -1$ and $V_c > 0$ the critical point $(x_c, 0, 0)$ is a stable local attractor for general cosmological solutions with barotropic matter. The corresponding geometry is the de Sitter spacetime with the scale factor $a(t) = \text{const} \exp[\sqrt{2V_c}\tau/(D-1)]$. Near the critical point $(x_c, 0, 0)$ the energy density behaves as $\epsilon \sim \exp[-b(1+w)\sqrt{2V_c}\tau]$, $\tau \rightarrow +\infty$, and it decays exponentially. Hence, the late time evolution of the corresponding models is governed by the effective cosmological constant determined by the value of the potential at its minimum. For the phantom matter, $w < -1$, one has $\lambda_3 > 0$ and the critical point $(x_c, 0, 0)$ is unstable. For the corresponding models the late time dynamics is driven by the non-gravitational matter. The qualitative analysis is more complicated for $V_c = 0$. In this case the critical point is degenerate and in order to determine the behaviour of the phase trajectories in its neighborhood one needs to keep nonlinear terms in the expansions of the right-hand sides of (47). The complete analysis of the models with a barotropic matter, including the points at the infinity of the phase space, will be discussed elsewhere.

4.2 Qualitative analysis in special cases

As an application of general analysis given above, first let us consider the Starobinsky model. The corresponding potential has the form (22) with $\phi/\phi_0 = x$. In the limit $x \rightarrow -\infty$ the potential behaves as $\exp[-(D+1)x/(D-1)]$. For the corresponding parameter σ one has $\sigma = (D+1)/(D-1)$ and, hence, $\sigma < 2b$. From here it follows that the point $\rho = 1$, $\theta = \pi/2$ is degenerate having an unstable node sector and a saddle sector (see figure 2). In the limit $x \rightarrow +\infty$ one has $V(x) \propto \exp[(D-3)x/(D-1)]$ and for $D > 3$ the behavior of the phase trajectories near the point $\rho = 1$, $\theta = 3\pi/2$ is similar to that for the point $\rho = 1$, $\theta = \pi/2$. In the special case $D = 3$ the dynamical system has a critical point at $\rho = 1$, $\theta = 0$. This point is a node (see the left panel in figure 2) and the corresponding unstable separatrix describes an inflationary expansion. This special solution is an attractor for the general solution. For $D = 3$ the point $\rho = 1$, $\theta = 3\pi/2$ at the infinity of the phase plane is an unstable node. The only critical point in the finite region of the phase plane, $(x, y) = (0, 0)$, corresponds to the minimum of the potential. This point is a stable sink and the corresponding geometry is the Minkowski spacetime. The phase portrait, mapped on the unit circle with the help of (61), is presented in the left panel of figure 2 for $D = 3$ and in the right panel for $D > 3$.

For the model (28) with an even n and $\beta_n > 0$, the potential has the form (29) with $\phi/\phi_0 = x$. For $n > (D+1)/2$ one has $V(x) \rightarrow 0$ in the limit $x \rightarrow +\infty$ (see graph (a) in the right panel of figure 1). In this case we have two critical points in the finite region of the phase plane. The first one, $(x, y) = (0, 0)$, corresponds to the minimum of the potential and is a stable sink. The second one, $(x, y) = (x_c, 0)$, corresponds to the maximum of the potential and is a saddle. The phase portrait is depicted in the left panel of figure 3. At infinity of the phase plane, the nature of the point $\rho = 1$, $\theta = \pi/2$ remains the same as in the previous example, whereas the point $\rho = 1$, $\theta = 3\pi/2$ becomes an unstable node. In the region $x \gg 1$, the potential is approximated by an exponential one, (51), with $\sigma = 1/(n-1) - 2/(D-1)$. For $n > (D+1)/2$ one has $\sigma < 0$ and the special solution with the asymptotic behavior (55) in the limit $t \rightarrow +\infty$ is an attractor for a general solution. Note that for the corresponding value of the parameter β_E we have

$$\beta_E = \left[\frac{2\sqrt{D}(n-1)}{2n-D-1} \right]^2 > 1, \quad (63)$$

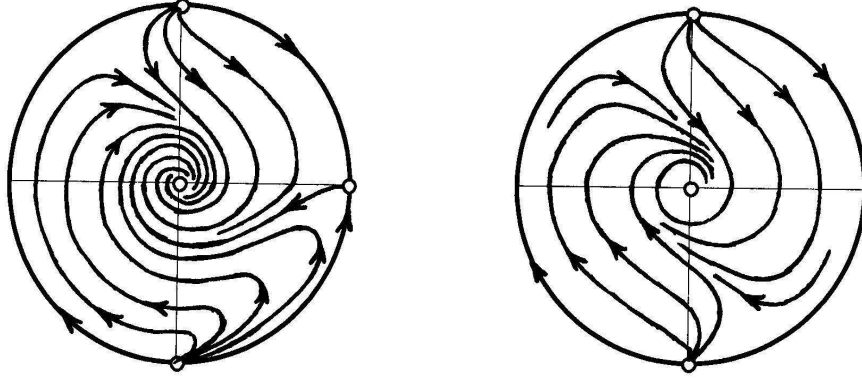


Figure 2: Phase portraits of the dynamical system for a $(D + 1)$ -dimensional generalization of the Starobinsky model in the cases $D = 3$ (left panel) and $D > 3$ (right panel).

and the solution (55) describes a late-time power-law inflation.

On the base of the general analysis in the previous subsection we can also plot the phase portraits for a general polynomial function (23). The asymptotics of the corresponding potential in the regions $\phi \rightarrow +\infty$ and $\phi \rightarrow -\infty$ are given by (26) and (27), respectively. If the coefficients of the exponents in these asymptotics are positive, the phase portraits in the regions $\phi \rightarrow \pm\infty$ are qualitatively equivalent to the one given on the right panel of figure 2 in the case $n < (D+1)/2$, and to the one on the left panel for $n = (D+1)/2$. In the case $n > (D+1)/2$, the phase portraits at the infinity of the phase plane is similar to that plotted on the left panel of figure 3. If the coefficients in the asymptotic expressions (26) and (27) are negative the potential goes to $-\infty$ in the limits $\phi \rightarrow \pm\infty$. In this case the regions of the phase plane near the points $\rho = 1, \theta = 0$ and $\rho = 1, \theta = \pi$ are classically forbidden and one has the transition from the expanding models to the contracting ones at the border of the forbidden region, similar to the one described on the right panel of figure 3 (see below). For a general polynomial function (23), instead of a single minimum, the corresponding potential can have a set of local minima in the finite region of the phase plane. In this case, the corresponding phase portrait is divided into regions which are separated by the stable separatrices of the saddles corresponding to neighboring local maxima of the potential. If the value of the potential at the minimum between these maxima is nonnegative, then one has a stable critical point corresponding to this minimum and it is an attractor (in the limit $t \rightarrow +\infty$) for all the trajectories between the stable separatrices of the neighboring saddles. Depending on the value of the potential at the local minimum the corresponding critical point can be either a stable sink or a stable node. If the value of the potential at the minimum is negative, then there is a classically forbidden region in the phase space (x, y) . This region is determined by the inequality

$$y^2 + 2V(x) < 0. \quad (64)$$

At the boundary of the forbidden region, given by $y^2 + 2V(x) = 0$, one has $H = 0$ and $\dot{H} = m_P^{1-D} V_E(\phi)/(D-1) < 0$. Hence, at the boundary the expansion stops at a finite value of the cosmological time t and then the model enters the stage of the contraction ($H < 0$). The corresponding dynamics is described by the dynamical system (47) with the opposite sign of the first term in the right-hand side of the second equation. Note that for nonnegative potentials the expansion-contraction transition in models with flat space is not classically allowed.

For the function (35) with $f_0\gamma = -m_P^{D-1}$ the potential is given by the expression (36). In the case $\gamma < 0$ the qualitative behavior of this potential is similar to that for the function (28) with $n > (D+1)/2$ and the corresponding phase portrait is qualitatively equivalent to the one presented

in the left panel of figure 3. However, note that the asymptotic behavior of the potential in the limit $\phi \rightarrow +\infty$ is not purely exponential. For the corresponding potential in (47) from (36) one has $V(x) \approx V_2 x e^{-2x/(D-1)}$ in the limit $x \gg 1$. Here, V_2 is expressed in terms of the coefficient V_{E0} in (36). It can be seen that the dynamical system has a special solution with the asymptotic behavior $y^2 \approx 2V(x)/(D^2 - 1)$ in the region $x \gg 1$ (compare with the special solution (54) for the case of pure exponential potentials). This special solution is an attractor for the general solution near the critical point $(\rho, \theta) = (1, 0)$. The corresponding time dependence of the scalar field is determined from the relation $2V(x) \approx (D-1)^2 (D^2 - 1)/\tau^2$, which is obtained by the integration of the first equation in (47). With the help of this relation, the asymptotic behavior of the Einstein frame scale factor is found from (49): $a(t) \approx a_0(t/t_0)^D$, $t \rightarrow +\infty$. The asymptotic behavior of x near the critical point $(\rho, \theta) = (1, 0)$, as a function of the time coordinate is simpler in the Jordan frame. By using the expression for the function $y(x)$ and the relation $dt_J = e^{-x/(D-1)} dt$, in the region $x \gg 1$ we can see that $x \approx V_2 \tau_J^2 / [2(D^2 - 1)]$, where $\tau_J = t_J/t_0$. For the scale factor in the Jordan frame one gets $a_J(t_J) \approx \text{const } e^x$.

In the case of the function (32) with $f_0 \gamma = -m_P^{D-1}$, the potential is given by the expression (34). For $\gamma > 0$ and in the limit of small curvatures, this model reduces to General Relativity with a negative cosmological constant. A characteristic feature of the potential is the presence of the region in the field space where it is negative. For this type of potentials there is a classically forbidden region determined by (64). As it has been noted above, at the boundary of this region the expansion stops at a finite value of the cosmological time t and then the model enters the stage of the contraction. For the potential (34), the only critical points of the dynamical system (47) are at the infinity of the phase plane. The corresponding phase portrait is depicted in the right panel of figure 3. The classically forbidden region of the phase space is shaded. The full/dashed trajectories correspond to the expansion/contraction phases. As it follows from (47), the trajectories for the contraction stage are obtained from those describing an expansion by the transformation $\tau \rightarrow -\tau$, $y \rightarrow -y$. For expanding models, near the point $\rho = 1$, $\theta = \pi/2$ the phase portrait has two sectors: an unstable node sector and a saddle sector. The point $\rho = 1$, $\theta = 3\pi/2$ is an unstable node. Depending on the initial conditions, the expanding models start their evolution at finite cosmic time $t = t_i$ from the point $\rho = 1$, $\theta = \pi/2$ or from the point $\rho = 1$, $\theta = 3\pi/2$. During a finite time interval the trajectories reach the boundary of the forbidden region (64) at $t = t_c > t_i$. At this moment the expansion stops ($H(t_c) = 0$) and the model enters the contraction stage (dashed trajectories on the phase portrait). The corresponding trajectories enter the critical points $\rho = 1$, $\theta = 3\pi/2$ and $\rho = 1$, $\theta = \pi/2$ at finite time $t_f > t_c$. Hence, all the models have a finite lifetime $t_f - t_i$.

For the Jordan frame Hubble function, from (46) for the gravi-scalar models one gets

$$H_J = \frac{e^{x/(D-1)}}{(D-1)t_0} [\pm \sqrt{y^2 + 2V(x)} - y]. \quad (65)$$

The corresponding comoving time coordinate is determined from the relation $dt_J = e^{-x/(D-1)} dt$. Similar to the Einstein frame, for nonnegative potentials, $V(x) \geq 0$, the expansion and the contraction models in the Jordan frame are separated by a classically forbidden region. For potentials with a region where $V(x) < 0$, for the models with the initial expansion the Jordan frame Hubble function vanishes at the points of the phase space (x_c, y) , where x_c is the zero of the potential $V(x)$, $V(x_c) = 0$. The expansion to contraction transition occurs at these points. The equation of the transition curve on the (ρ, θ) plane, defined in accordance with (61), is given by $\rho = 1/[1 + \cos(\theta)/x_c]$. For the example presented in the right panel of figure 3, compared with the Einstein frame, the expansion to contraction transition in the Jordan frame occurs earlier (later) for the expansion trajectories originating from the point $\rho = 1$, $\theta = \pi/2$ ($\rho = 1$, $\theta = 3\pi/2$).

As an example of comparison of the cosmological dynamics in different frames let us consider the special solution (55) for the exponential potential (51) in the Jordan frame. In the case $\sigma \neq -2/(D-1)$,

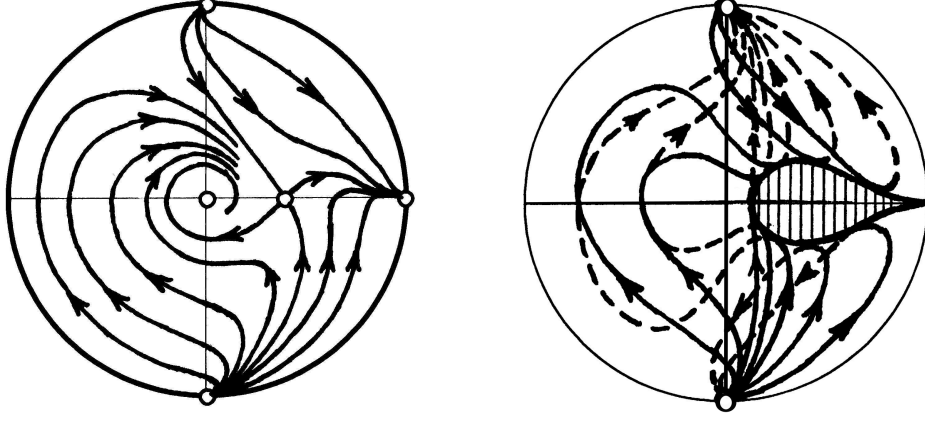


Figure 3: Phase portraits of the dynamical system for the potential (29), with $n = 4$, and for the potential (34).

for the corresponding cosmic time one gets

$$t_J = \frac{t_0 \tau^{u_0}}{u_0 x_1^{1/(D-1)}}, \quad u_0 \equiv \frac{2}{\sigma(D-1)} + 1. \quad (66)$$

As is seen, we have $0 < t_J < \infty$ for $u_0 > 0$ and $-\infty < t_J < 0$ for $u_0 < 0$. For the scalar field and the scale factor in the Jordan frame we find the expressions

$$\begin{aligned} \varphi(t_J) &= m_P^{D-1} x_1^b (u_0 t_J / t_0)^{-2/(\sigma u_0)}, \\ a_J(t_J) &= \text{const} (u_0 t_J / t_0)^{\beta_J}, \quad \beta_J = \frac{1 + 2b/\sigma}{1 + \sigma(D-1)/2}. \end{aligned} \quad (67)$$

Note that $u_0 \beta_J = (u_0 - 1)^2 [D + \sigma(D-1)/2]$. By taking into account that the special solution under consideration is present for $|\sigma| < 2b$, we conclude that $u_0 \beta_J > 0$. Depending on the value of the parameter σ , we have three qualitatively different types of evolutions. In the region $-\infty < \sigma < -2/(D-1)$ one has $u_0, \beta_J > 0$, $0 < t_J < \infty$ and, hence, $a_J(t_J) \rightarrow +\infty$, $\varphi(t_J) \rightarrow +\infty$ for $t_J \rightarrow +\infty$. For $-2/(D-1) < \sigma < 0$ we get $u_0, \beta_J < 0$, $-\infty < t_J < 0$. In this case $a_J(t_J) \rightarrow +\infty$, $H_J \rightarrow \infty$, $\varphi(t_J) \rightarrow \infty$ in the limit $t_J \rightarrow 0$ and the point $t_J = 0$ corresponds to a future singularity of the "Big Rip" type (for recent discussions of different types of future cosmological singularities see [24]). For $\sigma > 0$ we have $u_0, \beta_J > 0$, $0 < t_J < \infty$ and $a_J(t_J) \rightarrow +\infty$, $\varphi(t_J) \rightarrow 0$ in the limit $t_J \rightarrow +\infty$.

For $\sigma = -2/(D-1)$ one has the relation

$$t_J = t_0 \frac{D(D-1)}{\sqrt{2V_1}} \ln \tau, \quad (68)$$

with $-\infty < t_J < +\infty$, and the special solution for the exponential potential (51) takes the form

$$\begin{aligned} \varphi(t_J) &= m_P^{D-1} x_1 \exp \left(\frac{\sqrt{2V_1}}{Dt_0} t_J \right), \\ a_J(t_J) &= \text{const} \exp \left(\frac{\sqrt{2V_1}}{Dt_0} t_J \right). \end{aligned} \quad (69)$$

In this case we have an exponential inflation in the Jordan frame.

For the model (28), the potential in the region $x \gg 1$ is approximated by an exponential one with $\sigma = 1/(n-1) - 2/(D-1)$. In this case for the Jordan frame parameters in (67) one has

$$u_0 = \frac{D-1}{D+1-2n}, \quad \beta_J = \frac{2(n-1)(2n-1)}{D+1-2n}. \quad (70)$$

For $n > (D+1)/2$ these parameters are negative and, hence, $-\infty < t_J < 0$. In this case we have a "Big Rip" type singularity at $t_J = 0$ in the Jordan frame. This is the case for the example presented in the left panel of figure 3. Note that, unlike to the Einstein frame, the trajectories of the general solution enter the critical point $(\rho, \theta) = (1, 0)$ at finite value of the Jordan frame time coordinate t_J .

5 Conclusion

In the present paper we have considered the qualitative evolution of cosmological models in $(D+1)$ -dimensional $f(R)$ gravity. In order to do that the model is transformed to an equivalent model described by a scalar-tensor theory with the action (4). By a conformal transformation one can present the theory in various representations. From the point of view of the description of the cosmological dynamics, the most convenient representation corresponds to the Einstein frame, in which the gravitational part of the action coincides with that for General Relativity. In this frame there is a direct interaction of the scalar field with a non-gravitational matter.

For homogeneous and isotropic cosmological models with flat space the dynamics is described by the set of equations (38). These equations can be presented in the form of a third order autonomous dynamical system (40). The corresponding phase space has an invariant subspace describing the graviscalar models in the absence of a non-gravitational matter. The dynamical system for these models is presented in the form (47). For a general case of the function $f(R)$, we have found the critical points of the system and their nature, including the points at the infinity of the phase plane. As applications of general analysis, various special cases of the function $f(R)$ are considered. As the first example, we have taken the $(D+1)$ -dimensional generalization of the Starobinsky model with a quadratic function $f(R)$. For $D = 3$ the corresponding phase portrait is depicted in the left panel of figure 2. In this case, the potential has a plateau in the limit $\phi \rightarrow +\infty$ which describes an inflationary expansion. The corresponding special solution, presented by the separatrix of the saddle point in the phase portrait, is an attractor of the general solution. The latter feature shows that the inflation is a general feature in these models. From the point of view of the Starobinsky inflation, the spatial dimension $D = 3$ is special: for $D > 3$ the inflationary attractor corresponding to the plateau of the potential is absent and the only stable critical point corresponds to the minimum of the potential (left panel of figure 2). The solution corresponding to the latter is Minkowski spacetime.

We have also considered a polynomial generalization of the Starobinsky model. In particular, in the model given by (28) with even n , the inflationary plateau is realized for $n = (D+1)/2$. For $n > (D+1)/2$ and in the limit $\phi \rightarrow +\infty$ the potential decays exponentially. In this region models with power-law inflation are realized. The corresponding phase portrait for $n = 4$ is depicted in the left panel of figure 3. Depending on the initial conditions two classes of cosmological models are realized. For the first one, presented by the phase trajectories on the left of the stable separatrices of the saddle point, corresponding to the maximum of the potential, the spacetime geometry tends to the Minkowski one in the limit $t \rightarrow +\infty$. For the models from the second class the future attractor is at infinity of the phase space (the critical point $(\rho, \theta) = (1, 0)$). The asymptotic behavior of the scale factor for $t \rightarrow +\infty$ is given by (55) with the power (63). It presents a power-law inflation in the Einstein frame. In the Jordan frame the corresponding asymptotic is given by (67) with the parameters (70) and describes a "Big Rip" type of singularity.

As further applications of general procedure, we have discussed two examples of the exponential function $f(R)$. For the first one $f(R) = -m_P^{D-1} e^{\gamma R}/\gamma$, $\gamma > 0$, and in the weak field limit the model reduces to General Relativity with a negative cosmological constant. The potential is given by (34)

and the phase portrait is presented in the right panel of figure 3. The shaded region corresponds to the classically forbidden region in the phase plane. This type of regions arise for potentials taking negative values in some range of the field space. At the boundary of the forbidden region the expansion stops and the model enters the contraction phase. For the second example $f(R) = -m_P^{D-1} (e^{\gamma R} - 1) / \gamma$, $\gamma < 0$, and the potential is given by (36). This potential is non-negative everywhere and has a minimum with a zero cosmological constant. The corresponding phase portrait is qualitatively equivalent to the one depicted in the left panel of figure 3. In this example the asymptotic behavior of the potential in the limit $\phi \rightarrow +\infty$ is not purely exponential. In this region one has a power-law inflation in the Einstein frame and the exponential inflation in the Jordan frame.

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